Frequency Response

Michel M. Maharbiz
Vivek Subramanian

Frequency Response

Transfer function of a circuit or system describes the output response to an input excitation as a function of the angular frequency ω.

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_{\text{out}}(\omega)}{\mathbf{V}_{\text{in}}(\omega)}$$ Voltage Gain

$$\mathbf{H}(\omega) = M(\omega)\, e^{j\phi(\omega)},$$

where by definition,

$$M(\omega) = |\mathbf{H}(\omega)|, \qquad \phi(\omega) = \tan^{-1}\left\{\frac{\mathfrak{Im}[\mathbf{H}(\omega)]}{\mathfrak{Re}[\mathbf{H}(\omega)]}\right\}$$

Magnitude ($M(\omega)$) Phase ($\phi(\omega)$)

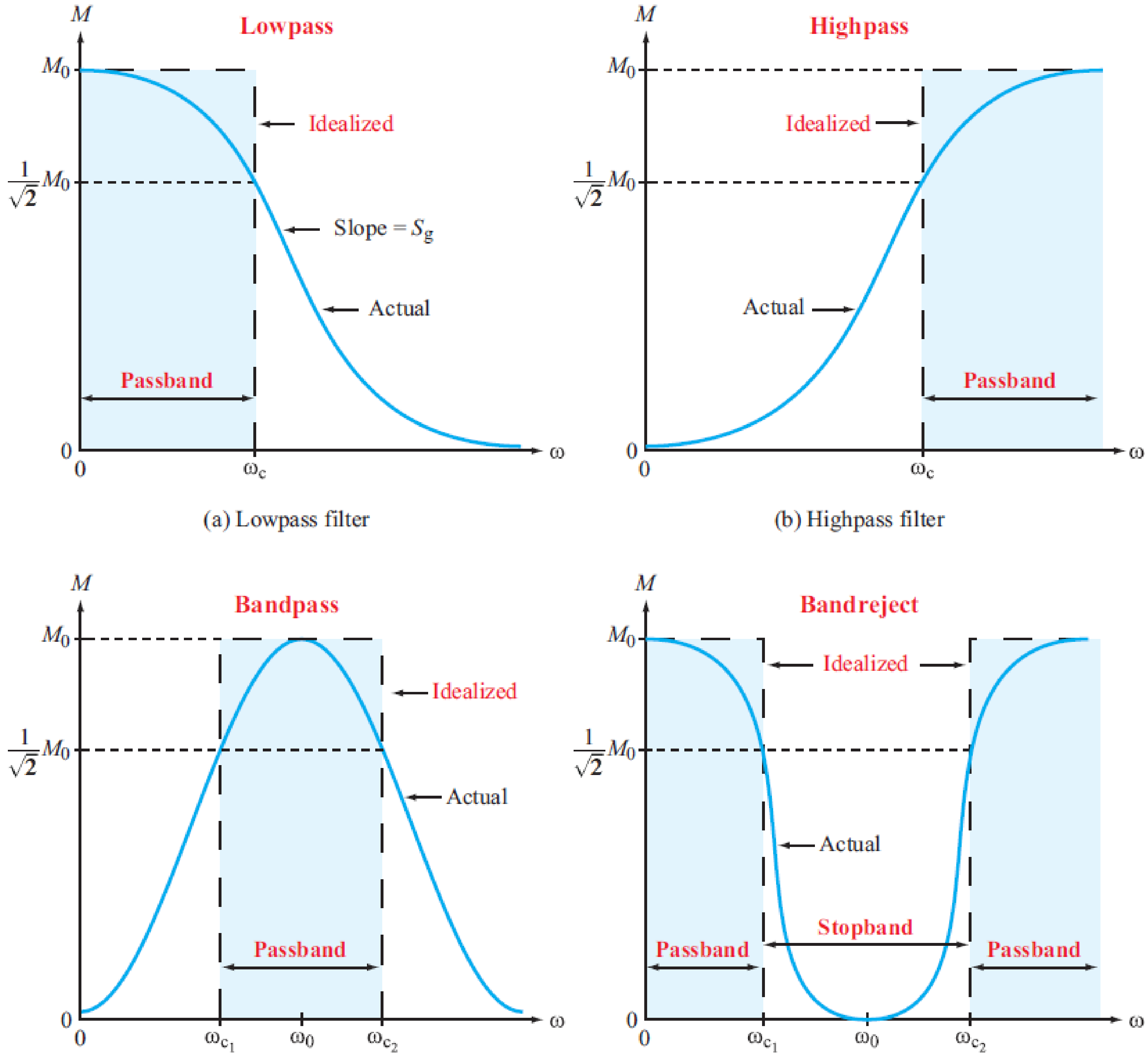
Other Transfer Functions

Current gain: $\mathbf{H}_I(\omega) = \dfrac{\mathbf{I}_{\text{out}}(\omega)}{\mathbf{I}_{\text{in}}(\omega)},$

Transfer impedance: $\mathbf{H}_Z(\omega) = \dfrac{\mathbf{V}_{\text{out}}(\omega)}{\mathbf{I}_{\text{in}}(\omega)},$

Transfer admittance: $\mathbf{H}_Y(\omega) = \dfrac{\mathbf{I}_{\text{out}}(\omega)}{\mathbf{V}_{\text{in}}(\omega)}.$

Examples



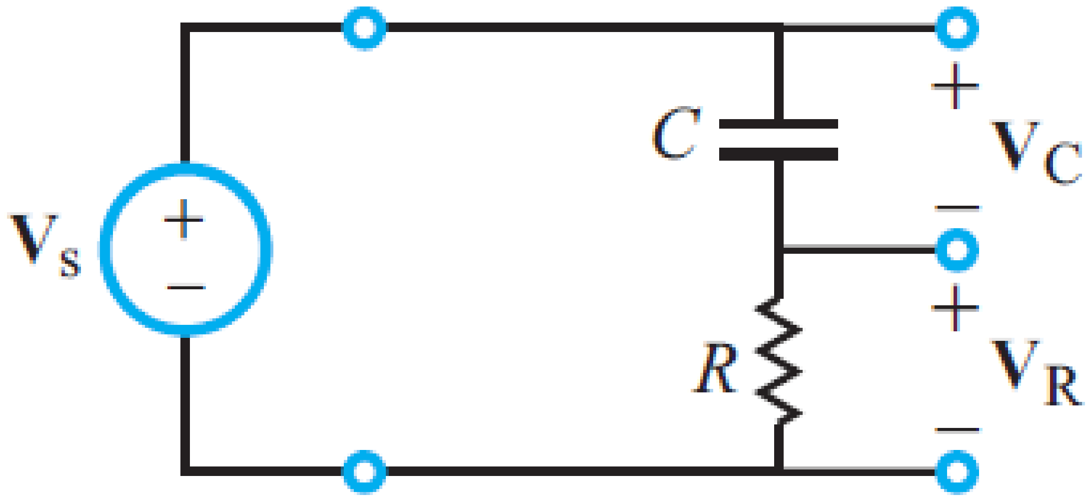
(a) Lowpass filter

(b) Highpass filter

(c) Bandpass filter

(d) Bandreject filter

Example: Low pass filter



(a) RC circuit

Application of voltage division gives

$$\mathbf{V}_C = \frac{\mathbf{V}_s \mathbf{Z}_C}{R + \mathbf{Z}_C} = \frac{\mathbf{V}_s / j\omega C}{R + \frac{1}{j\omega C}}.$$

The transfer function corresponding to $\mathbf{V}_C$ is

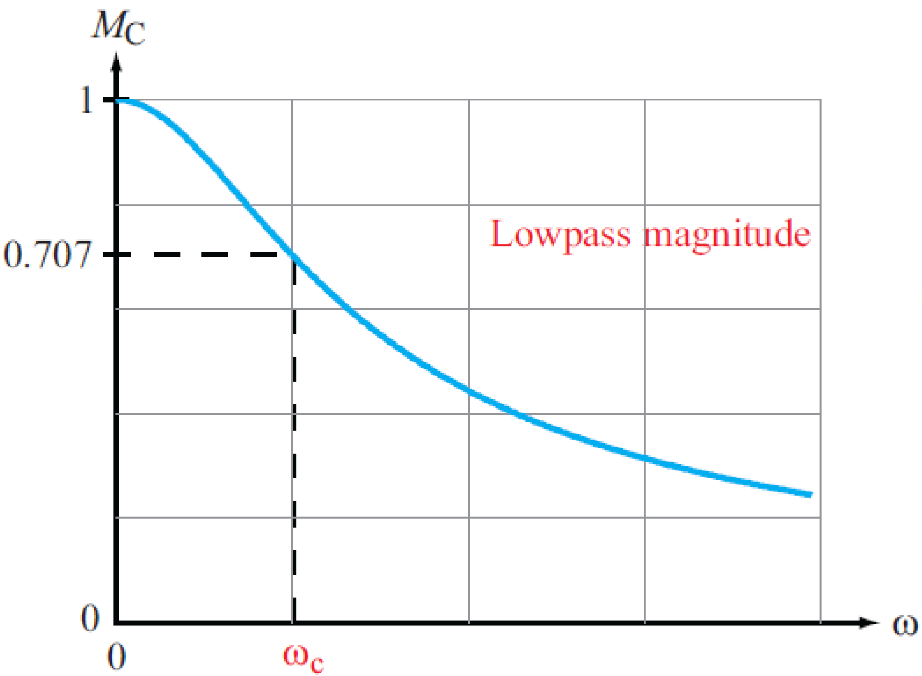
$$\mathbf{H}_C(\omega) = \frac{\mathbf{V}_C}{\mathbf{V}_s} = \frac{1}{1 + j\omega RC},$$

$$\mathbf{H}_C(\omega) = M_C(\omega)\, e^{j\phi_C(\omega)},$$

$$M_C(\omega) = |\mathbf{H}_C(\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\phi_C(\omega) = -\tan^{-1}(\omega RC).$$

Example: Low pass filter



$$\mathbf{H}_C(\omega) = M_C(\omega)\, e^{j\phi_C(\omega)},$$

$$M_C(\omega) = |\mathbf{H}_C(\omega)| = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}$$

$$\phi_C(\omega) = -\tan^{-1}(\omega RC).$$

$$\mathbf{H}_C(\omega) = \frac{\mathbf{V}_C}{\mathbf{V}_s} = \frac{1}{1+j\omega RC},$$

Corner Frequency ω_c

The corner frequency ω_c is defined as the angular frequency at which $M(\omega)$ is equal to $1/\sqrt{2}$ of the reference peak value,

$$M(\omega_c) = \frac{M_0}{\sqrt{2}} = 0.707 M_0. \tag{9.5}$$

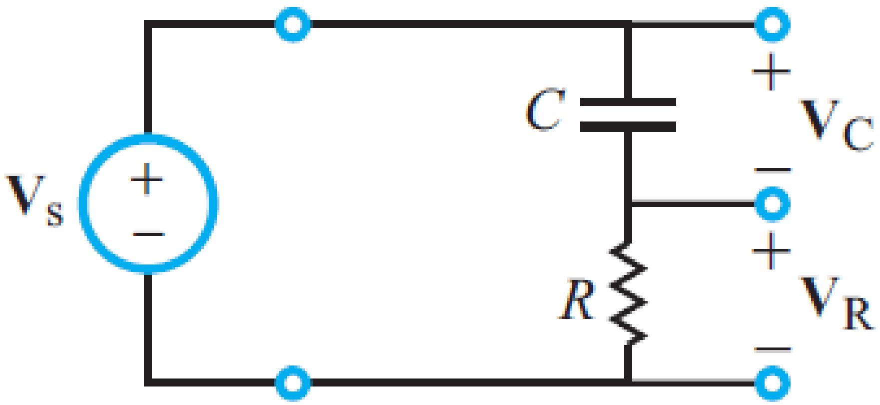
To determine corner frequency:

$$M_C^2(\omega_c) = \frac{1}{1+\omega_c^2 R^2 C^2} = \frac{1}{2},$$

leads to

$$\omega_c = \frac{1}{RC}.$$

Example: High pass filter



(a) RC circuit

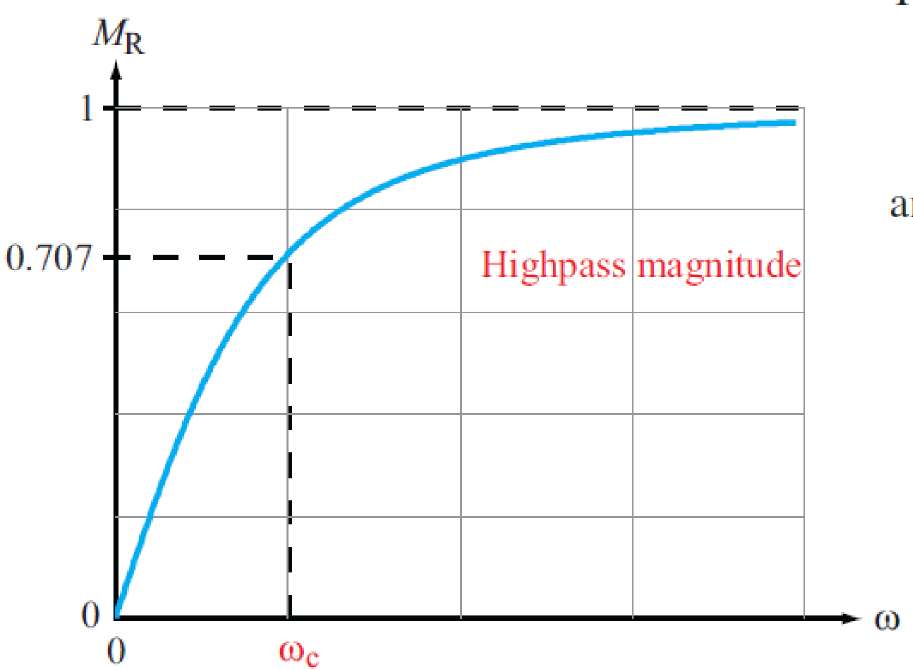
$$\mathbf{H}_R(\omega) = \frac{\mathbf{V}_R}{\mathbf{V}_s} = \frac{j\omega RC}{1 + j\omega RC}.$$

The magnitude and phase angle of $\mathbf{H}_R(\omega)$ are given by

$$M_R(\omega) = |\mathbf{H}_R(\omega)| = \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}$$

and

$$\phi_R(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega RC).$$



Another example: an RLC circuit

$$\mathbf{I} = \frac{\mathbf{V}_s}{R + j(\omega L - \frac{1}{\omega C})} = \frac{j\omega C \mathbf{V}_s}{(1-\omega^2 LC) + j\omega RC}$$

$$\mathbf{H}_{BP}(\omega) = \frac{\mathbf{V}_R}{\mathbf{V}_s} = \frac{R\mathbf{I}}{\mathbf{V}_s} = \frac{j\omega RC}{(1-\omega^2 LC) + j\omega RC}$$

$$M_{BP}(\omega) = |\mathbf{H}_{BP}(\omega)| = \frac{\omega RC}{\sqrt{(1-\omega^2 LC)^2 + \omega^2 R^2 C^2}}$$

$$\phi_R(\omega) = 90^\circ - \tan^{-1}\left[\frac{\omega RC}{1-\omega^2 LC}\right]$$

$$\omega_{c_1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}},$$

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

$$\omega_{c_2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}.$$

$$B = \omega_{c_2} - \omega_{c_1} = \frac{R}{L}.$$